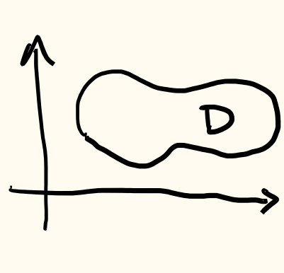
Meeting 12 Math 22

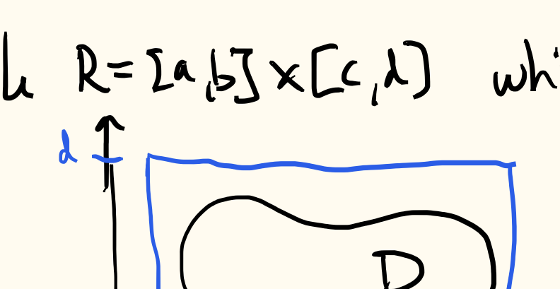
Section 11.3.1 Double Integrals over General Regions

Let D be a closed and bounded and let $f(x,y)$ be continuous on D. Define a new function



$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & \text{otherwise.} \end{cases}$$

Choose a rectangle $R = [a,b] \times [c,d]$ which contains D.



The double int. of $f(x,y)$ over D is defined as

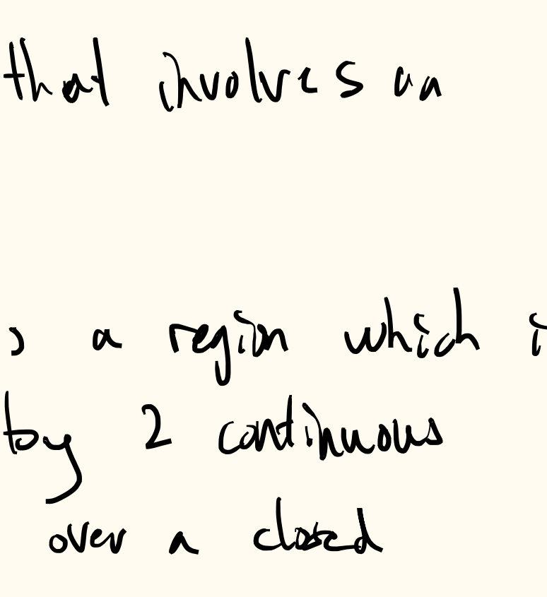
$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_c^d \int_a^b F(x,y)\,dx\,dy$$

The definition makes sense because $f(x,y) = F(x,y)$ on D and $F(x,y)$ contributes zero volume outside of D.

When the region D is "nice" we can find a better formula that involves an iterated integral.

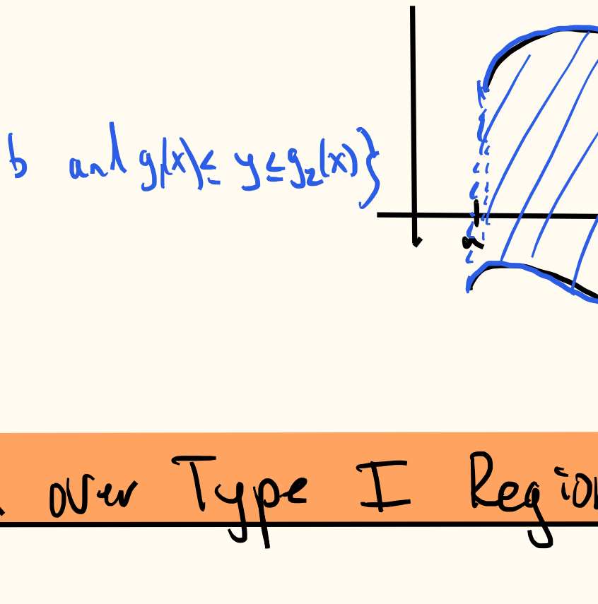
Type I Region Suppose D is a region which is bounded above and below by 2 continuous functions $g_1(x)$ and $g_2(x)$ over a closed interval $[a,b]$.

$D = \{(x,y) \mid a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$



Double Integral over Type I Region:

Choose a rectangle $R = [a,b] \times [c,d]$ which contains D



Then $\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$

$$= \int_a^b \int_c^d F(x,y)\,dy\,dx$$

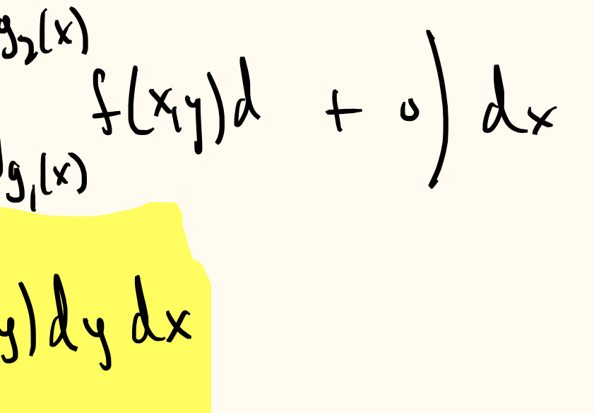
$$= \int_a^b \left(\int_c^{g_1(x)} F(x,y)\,dy + \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy + \int_{g_2(x)}^d F(x,y)\,dy \right) dx$$

$$= \int_a^b \left(0 + \int_{g_1(x)}^{g_2(x)} f(x,y)\,d + 0 \right) dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type II Suppose D is bounded by two continuous function $h_1(y)$ and $h_2(y)$ over an interval $[c,d]$.

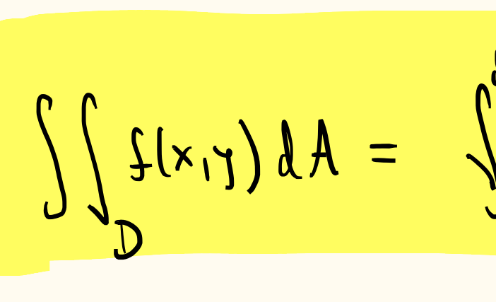
$D = \{(x,y) \mid c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$

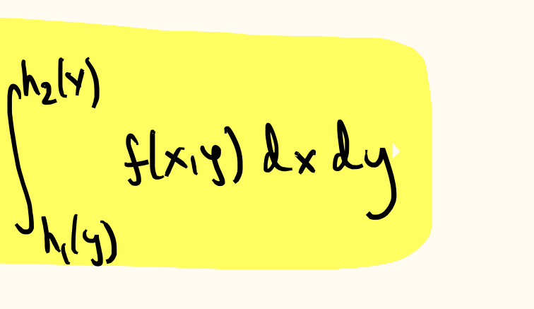


Double Integral over Type II Region:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

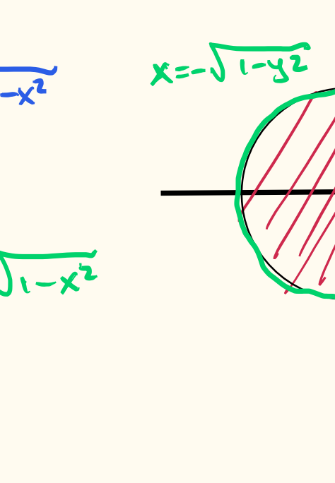
Example A region that is both Type 1 and Type 2





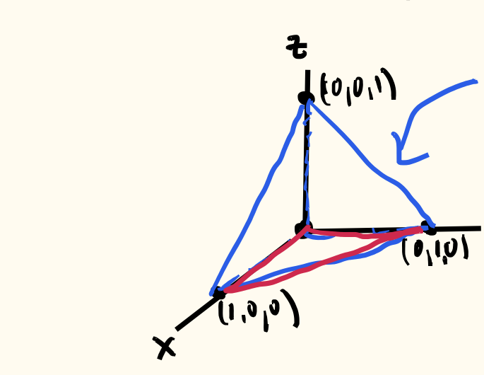
Example A region that is neither Type I or Type II

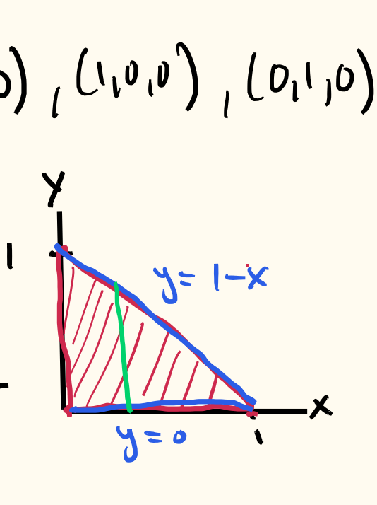
★ Type I/II regions do not have holes ★



Example Volume of Tetrahedron from Preview 11.3.1

Tetrahedron w/ vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$





As a Type I region

$$\iint_T 1-x-y\,dA = \int_0^1 \int_0^{1-x} 1-x-y\,dy\,dx$$

$$= \int_0^1 \left[y - xy - \tfrac{1}{2}y^2 \right]_0^{1-x} dx$$

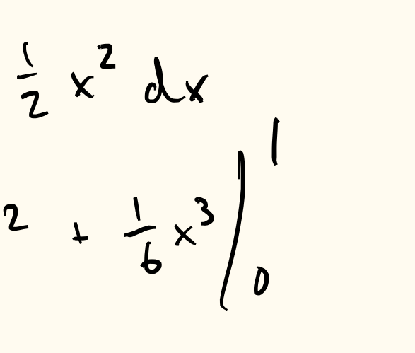
$$= \int_0^1 1-x-x(1-x) - \tfrac{1}{2}(1-x)^2\,dx$$

$$= \int_0^1 \tfrac{1}{2} - x + \tfrac{1}{2}x^2\,dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 \Big|_0^1$$

$$= \tfrac{1}{6}$$

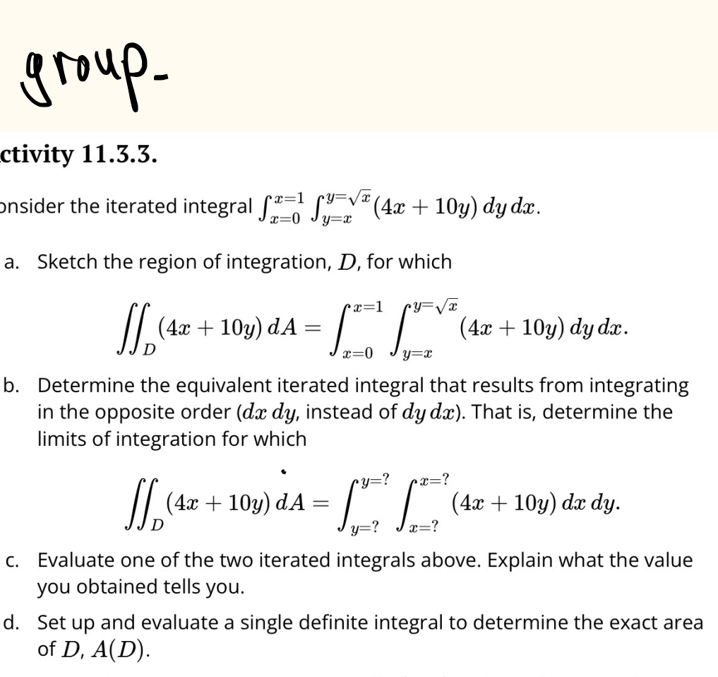
As a Type II region



$$\iint_T 1-x-y\,dA = \int_0^1 \int_0^{1-y} 1-x-y\,dx\,dy$$

Activity 11.3.3

- Complete w/ your group.
- Class discussion.



Activity 11.3.3.

Consider the iterated integral $\int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} (4x+10y)\,dy\,dx$.

a. Sketch the region of integration, D, for which

$$\iint_D (4x+10y)\,dA = \int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} (4x+10y)\,dy\,dx.$$

b. Determine the equivalent iterated integral that results from integrating in the opposite order ($dx\,dy$, instead of $dy\,dx$). That is, determine the limits of integration for which

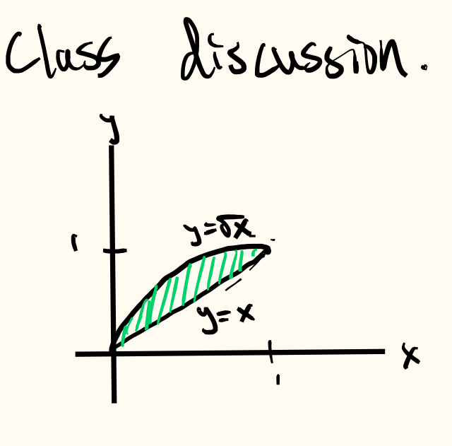
$$\iint_D (4x+10y)\,dA = \int_{y=?}^{y=?} \int_{x=?}^{x=?} (4x+10y)\,dx\,dy.$$

c. Evaluate one of the two iterated integrals above. Explain what the value you obtained tells you.

d. Set up and evaluate a single definite integral to determine the exact area of D, $A(D)$.

e. Determine the exact average value of $f(x,y) = 4x+10y$ over D.

(a)



(b) $\iint_D 4x+10y\,dA$

$$= \int_0^1 \int_{y^2}^{y} 4x+10y\,dx\,dy$$

(c)
$$= \int_0^1 \left[2x^2 + 10xy \right]_{y^2}^{y} dy$$

$$= \int_0^1 2y^2 + 10y^2 - 2y^4 - 10y^3\,dy$$

$$= \tfrac{2}{3}y^3 + \tfrac{10}{3}y^3 - \tfrac{2}{5}y^5 - \tfrac{10}{4}y^4 \Big|_0^1$$

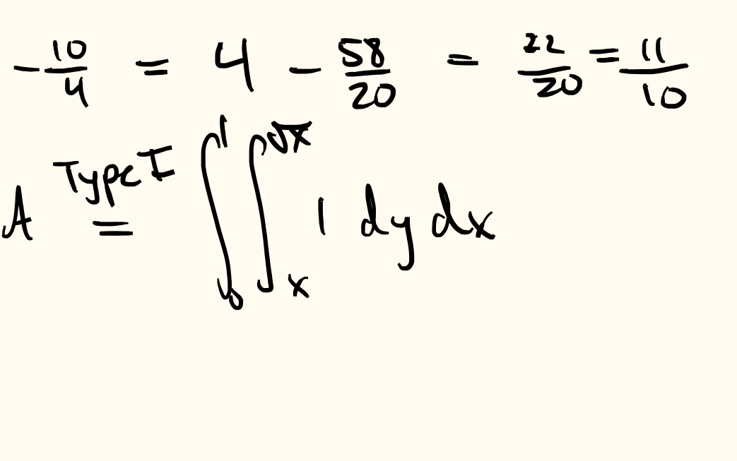
$$= \tfrac{2}{3} + \tfrac{10}{3} - \tfrac{2}{5} - \tfrac{10}{4} = 4 - \tfrac{58}{20} = \tfrac{22}{20} = \tfrac{11}{10}$$

(d) $A(D) = \iint_D 1\,dA \overset{\text{Type I}}{=} \int_0^1 \int_x^{\sqrt{x}} 1\,dy\,dx$

(e) Skip

Activity 11.3.4

- Complete as a class



Activity 11.3.4.

Consider the iterated integral $\int_{x=0}^{x=4} \int_{y=x/2}^{y=2} e^{y^2}\,dy\,dx$.

a. Explain why we cannot find a simple antiderivative for e^{y^2} with respect to y, and thus are unable to evaluate $\int_{x=0}^{x=4} \int_{y=x/2}^{y=2} e^{y^2}\,dy\,dx$ in the indicated order using the Fundamental Theorem of Calculus.

b. Given that $\iint_D e^{y^2}\,dA = \int_{x=0}^{x=4} \int_{y=x/2}^{y=2} e^{y^2}\,dy\,dx$, sketch the region of integration, D.

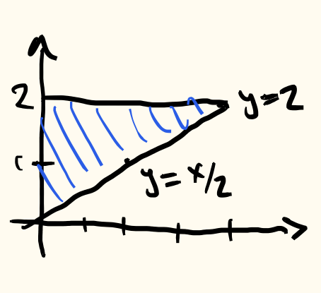
c. Rewrite the given iterated integral in the opposite order, using $dA = dx\,dy$. (Hint: You may need more than one integral.)

d. Use the Fundamental Theorem of Calculus to evaluate the iterated integral you developed in (c). Write one sentence to explain the meaning of the value you found.

e. What is the important lesson this activity offers regarding the order in which we set up an iterated integral?

(a) There is no elementary antiderivative.

(b)



The region is Type I and Type II.

(c) $\iint_D e^{y^2}\,dA = \int_0^2 \int_0^{2y} e^{y^2}\,dx\,dy$

(d) $= \int_0^2 x e^{y^2} \Big|_0^{2y} dy$

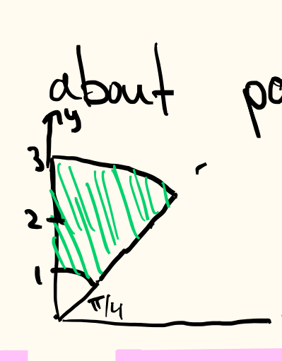
$= \int_0^2 2y e^{y^2}\,dy$ $\quad u = y^2 \quad du = 2y\,dy$

$= \int_{0=u(0)}^{4=u(2)} e^u\,du = e^4 - 1.$

Section 11.5 Double Integrals in Polar Coordinates

Reading Debrief

- Discuss Activity 11.5.2 w/ the person next to you
- Questions about polar coordinates?



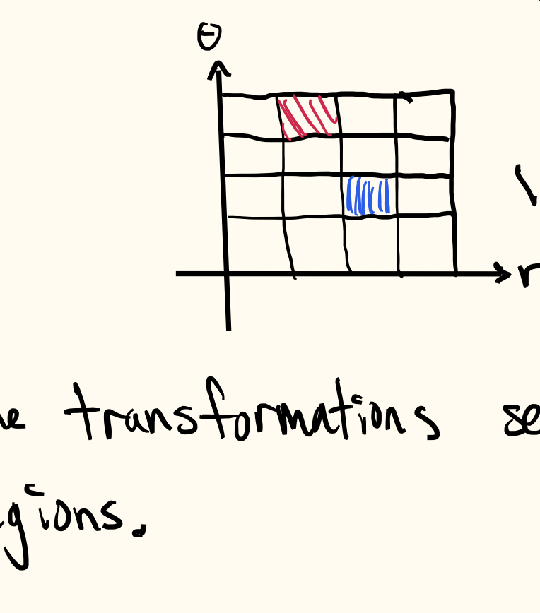
$\pi/4 \le \theta \le \pi/2$
$1 \le r \le 3$

Section 11.5.2 Integrating in Polar

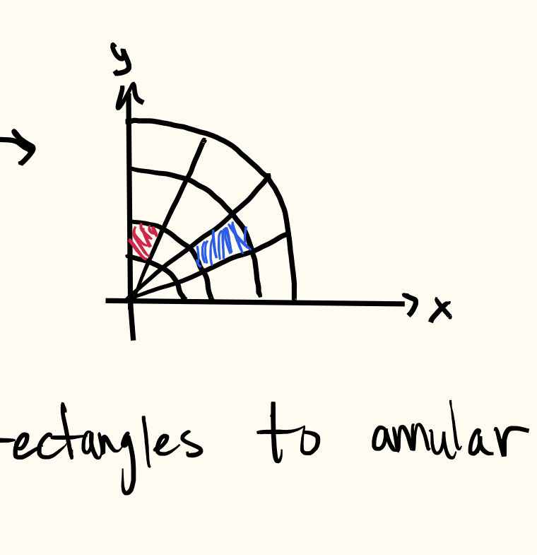
The equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ can be thought of as a function $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

The function Φ transforms the "cartesian plane" into the "polar plane"



$\xrightarrow{\Phi}$



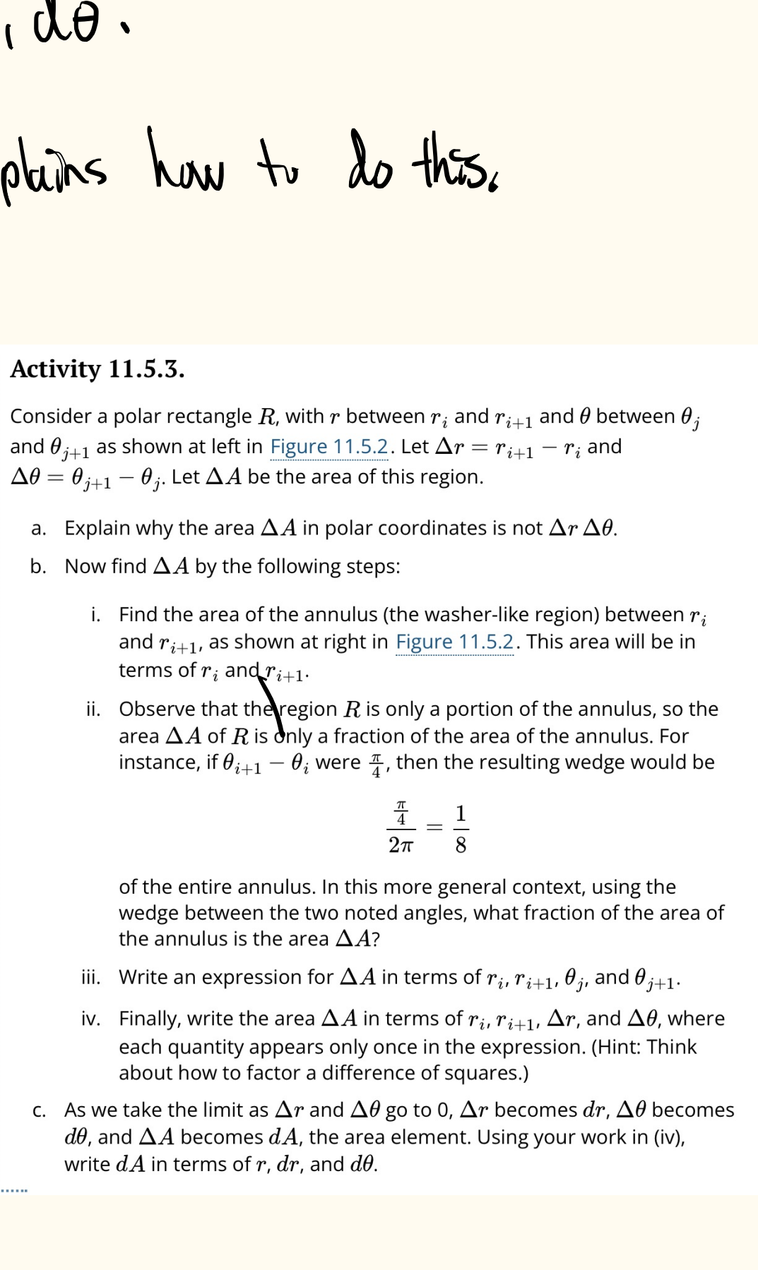
The transformations sends rectangles to annular regions.

We can use polar coordinates to evaluate double integrals like $\iint_R e^{x^2+y^2}\,dA$ where R is a circle or an annulus or a "circular region". The idea is to make the substitution $x = r\cos\theta$ and $y = r\sin\theta$ everywhere. However, Φ does not preserve area, so we must calculate $dA = dx\,dy$ in terms of $r, dr, d\theta$.

The next activity explains how to do this.

Activity 11.5.3

- Together as a class.



Activity 11.5.3.

Consider a polar rectangle R, with r between r_i and r_{i+1} and θ between θ_j and θ_{j+1} as shown at left in Figure 11.5.2. Let $\Delta r = r_{i+1} - r_i$ and $\Delta\theta = \theta_{j+1} - \theta_j$. Let ΔA be the area of this region.

a. Explain why the area ΔA in polar coordinates is not $\Delta r\,\Delta\theta$.

b. Now find ΔA by the following steps:

   i. Find the area of the annulus (the washer-like region) between r_i and r_{i+1}, as shown at right in Figure 11.5.2. This area will be in terms of r_i and r_{i+1}.

   ii. Observe that the region R is only a portion of the annulus, so the area ΔA of R is only a fraction of the area of the annulus. For instance, if $\theta_{i+1} - \theta_i$ were $\frac{\pi}{4}$, then the resulting wedge would be

   $$\frac{\frac{\pi}{4}}{2\pi} = \frac{1}{8}$$

   of the entire annulus. In this more general context, using the wedge between the two noted angles, what fraction of the area of the annulus is the area ΔA?

   iii. Write an expression for ΔA in terms of r_i, r_{i+1}, θ_j, and θ_{j+1}.

   iv. Finally, write the area ΔA in terms of r_i, r_{i+1}, Δr, and $\Delta\theta$, where each quantity appears only once in the expression. (Hint: Think about how to factor a difference of squares.)

c. As we take the limit as Δr and $\Delta\theta$ go to 0, Δr becomes dr, $\Delta\theta$ becomes $d\theta$, and ΔA becomes dA, the area element. Using your work in (iv), write dA in terms of r, dr, and $d\theta$.

(b)

(i) Area of annulus is $A = \pi r_{i+1}^2 - \pi r_i^2$

(ii) The ratio is

$$\frac{\Delta A}{A} = \frac{\theta_{i+1} - \theta_i}{2\pi}$$

(iii) Using (i) and (ii)

$$\Delta A = \left(\pi r_{i+1}^2 - \pi r_i^2\right) \frac{(\theta_{i+1} - \theta_i)}{2\pi} = \frac{\pi(r_{i+1}+r_i)(r_{i+1}-r_i)\Delta\theta}{2\pi}$$

(iv) $= \dfrac{(r_{i+1}+r_i)\,\Delta r\,\Delta\theta}{2}$

(c) As $\Delta r \to 0$, $r_{i+1} \to r_i$. So

$$dA = \frac{2r\,dr\,d\theta}{2} = r\,dr\,d\theta$$

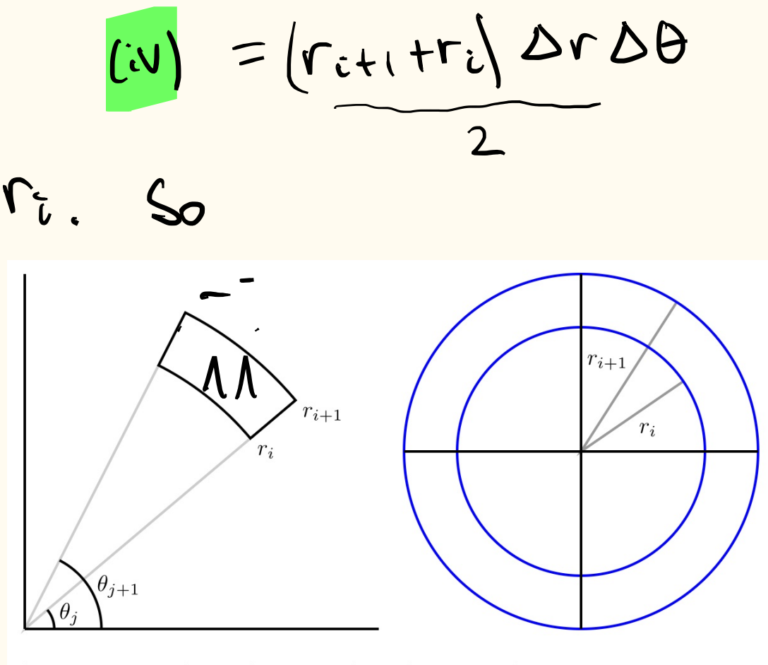


Figure 11.5.2. Left: A polar rectangle. Right: An annulus.

Since we know $dA = r\,dr\,d\theta$, we obtain:

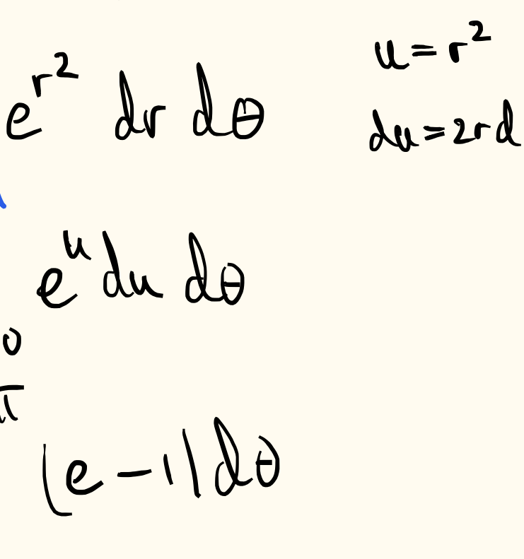
Double Integral in Polar Coordinates

$$\iint_D f(x,y)\,dA = \iint_{D'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where D' is a region whose image under Φ is equal D.

Example $f(x,y) = e^{x^2+y^2}$ on D = unit disk.

$\xrightarrow{\Phi}$



D' is a rectangle.

$$\iint_D e^{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^1 r e^{r^2}\,dr\,d\theta \qquad u = r^2 \quad u(1) = 1 \quad du = 2r\,dr \quad u(0) = 0$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^1 e^u\,du\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (e-1)\,d\theta$$

$$= \pi(e-1).$$